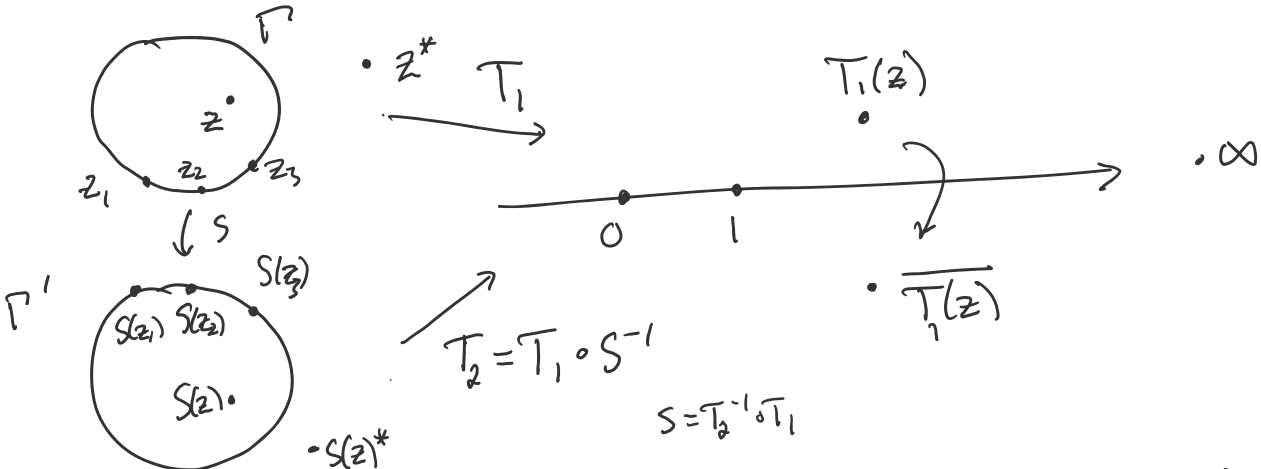
Lecture 16

Wednesday, November 6, 2019 5:47 AM

- Definition of symmetric pts from end of Lecture Notes 15 + note $z^{**} = z$.

<u>Thm1</u> (Symmetry principle). If MT S sends "circle" Γ to Γ' and z, z^* are symmetric w.r.t. Γ, then $S(z), S(z^*)$ are symmetric w.r.t. Γ', i.e. $S(z)^* = S(z^*)$.

<u>Pf.</u>



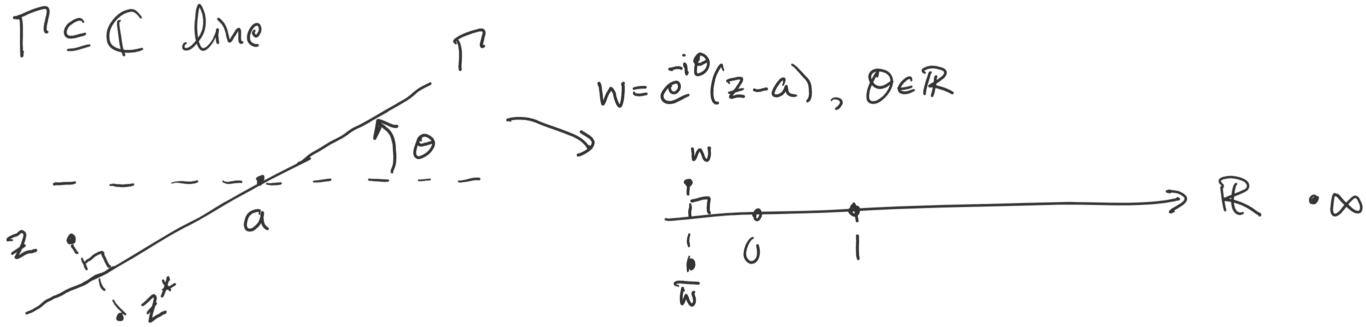
Pick $z_1, z_2, z_3 \in \Gamma$, let T_1 be map to $\{1, 0, \infty\}$. Let T_2 be map sending $S(z_1), S(z_2), S(z_3)$ to $\{1, 0, \infty\}$. Then $T_2 = T_1 \circ S^{-1}$. Thus, $S(z^*) = T_2^{-1}\left(\overline{T_1(z)}\right)$ and

$$S(z)^* = T_2^{-1}\left(\overline{T_2(S(z))}\right) = T_2^{-1}\left(\overline{(T_1 \circ S^{-1})(S(z))}\right) = T_2^{-1}\left(\overline{T_1(z)}\right) = S(z^*).$$

$\square$

<u>Geometric meaning of symmetric:</u>

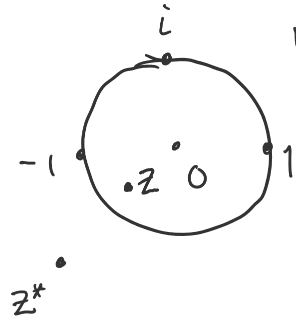
① $\Gamma \subseteq \mathbb{C}$ line

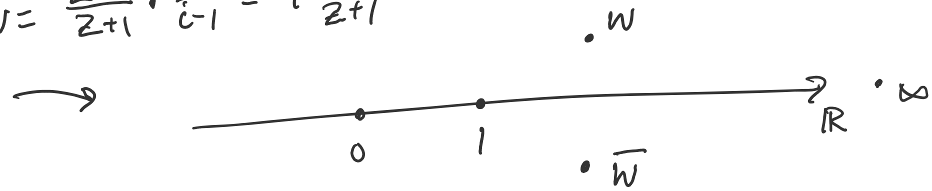


② Consider $\Gamma = \{|z| = 1\}$ unit circle.

i $\quad w = \frac{z-1}{z+1} \cdot \frac{i+1}{i-1} = -i \cdot \frac{z-1}{z+1}$ $\quad$. w

$$w = \frac{z-1}{z+1} \cdot \frac{i+1}{i-1} = -i \cdot \frac{z-1}{z+1}$$



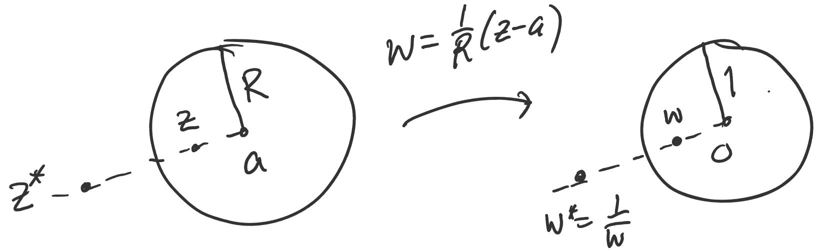


$$-i\,\frac{z^*-1}{z^*+1} = \overline{-i\,\frac{z-1}{z+1}} = i\,\frac{\bar z-1}{\bar z+1} \Rightarrow \underbrace{(z^*-1)(\bar z+1)}_{z^*\bar z + \cancel{z^*} - \cancel{\bar z} - 1} = \underbrace{-(z^*+1)(\bar z-1)}_{-(z^*\bar z - \cancel{z^*} + \cancel{\bar z} - 1)}$$

$$\Rightarrow 2z^*\bar z = 2 \;; \text{ i.e. } \boxed{z^* = 1/\bar z} \quad (\text{w/ } 0^* = \infty,\ \infty^* = 0)$$

<u>Note</u>: $z = re^{i\theta} \Rightarrow z^* = \frac{1}{r}e^{i\theta}$. Same ray, reciprocal radii.

③ In general, use symmetry principle; $\Gamma = \{|z-a| = R\}$.



$$\Rightarrow \frac{1}{R}(z^*-a) = \frac{1}{\frac{1}{R}(\bar z - \bar a)} \Rightarrow$$

$$z^* = \frac{R^2}{\bar z - a} + a$$; Again, same ray (from a) and "reciprocal" distances from a.

$$r \leftrightarrow \frac{R^2}{r}$$